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An enclosure method of eigenvalues for the elliptic operator linearized at an exact solution of nonlinear problems

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Abstract

We consider eigenvalue enclosing of the elliptic operator which is linearized at an exact solution of certain nonlinear elliptic equation. This problem is important in the mathematically rigorous analysis of the stability or bifurcation of some solutions for nonlinear problems. We formulate such a kind of eigenvalue problem as the nonlinear system which contains both linearized eigenvalue problem and the original nonlinear equation. We also consider the indices of eigenvalues, especially the first eigenvalue of such a problem. In these enclosing procedures, the finite-dimensional verified computations for linear and nonlinear system of equations play an essential role. A numerical example is presented. © 2001 Elsevier Science Inc. All rights reserved.

Keywords: Eigenvalue enclosing; Linearized eigenvalue problem; Nonlinear elliptic PDE.

1. Introduction

Several enclosure methods have been proposed for infinite-dimensional eigenvalue problems, and in particular the eigenvalues of elliptic operators (cf. [3,16,17]). We also proposed an enclosure method, which is different from these existing approaches, by applying our verification method for nonlinear PDEs [9,15]. In par-

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ticular, we considered the eigenvalue problems for operators linearized at an approximate solution of a nonlinear elliptic PDE. Also, in [8], using the estimation of eigenvalue which has minimum absolute value, we enclosed the exact solution of original nonlinear PDEs.

However, in the case that we treat the bifurcation phenomena or stabilities of solution, it is necessary to consider the exact eigenvalues of operators linearized at the "exact" solution of nonlinear problems instead of approximate solutions.

In principle, it is possible to enclose such an eigenvalue using the existing method [9,15] by replacing approximate solution by a set of functions satisfying verification condition for the original nonlinear problem. However, it is actually difficult to carry out the verification process, a kind of infinite-dimensional interval Newton method, due to the influence of the error accumulation.

In this paper, in order to overcome this difficulty, we consider a nonlinear system which contains both eigenvalue problem for linearized operator and original nonlinear problem, and then apply our method to this system. Considering the "triple" of exact solution of original nonlinear problem, eigenvalue and eigenfunction for linearized operator, we succeeded in verification of desired solution using infinite-dimensional Newton method. We will describe its technique in Section 2.

Although such a verified triple is locally unique as "triple", for the uniquness of each element of the triple, e.g., the uniqueness of eigenvalue in the obtained interval, we need another consideration. In Section 3, we prove that we can confirm the local uniqueness separately of eigenvalues and eigenfunctions by the method used in Section 2 as well as the uniqueness of solution for nonlinear PDE.

Since the minimum eigenvalue is important for the linearized operator at the exact solution of nonlinear PDE, we will describe it in Section 4. This is based on a sort of comparison theorem.

In Section 5, a numerical example is presented.

Particular emphasis is that the finite-dimensional verified computations for linear and nonlinear system of equations play an essential role in our enclosure method.

2. Verification method using nonlinear system

2.1. Problem and fixed point formulation

Let Ω be a bounded convex domain in $\mathbb{R}^2$. In what follows, for some integer m, let $H^m(\Omega)$ denote the L^2-Sobolev space of order m on Ω. Then, define $H_0^1(\Omega) \equiv \{v \in H^1(\Omega) \mid v = 0 \text{ on } \partial\Omega\}$ with the inner product $\langle u, v\rangle_{H_0^1} \equiv (\nabla u, \nabla v)_{L^2}$ for $u, v \in H_0^1(\Omega)$, and the norm $\|u\|_{H_0^1} \equiv \|\nabla u\|_{L^2}$ for $u \in H_0^1(\Omega)$, where $(\cdot,\cdot)_{L^2}$ and $\|\cdot\|_{L^2}$ represent the inner product and the norm on $L^2(\Omega)$, respectively. Also, we define the space X by $X \equiv H_0^1(\Omega) \times H_0^1(\Omega) \times \mathbb{R}$.

Now, let S_h be a finite-dimensional subspace of $H_0^1(\Omega)$ that depends on h $(0 < h < 1)$. Usually, S_h is taken to be a finite element subspace with mesh size h and define $X_h \equiv S_h \times S_h \times \mathbb{R}$. Also, let $P_{h0} : H_0^1(\Omega) \longrightarrow S_h$ denote the H_0^1-projection defined by

$$(\nabla(u - P_{h0}u), \nabla v)_{L^2} = 0 \quad \text{for all } v \in S_h.$$

We now assume the following approximation property in S_h.

Assumption 1. For any $u \in H^2(\Omega) \cap H_0^1(\Omega)$,

$$\inf_{\chi \in S_h} \|u - \chi\|_{H_0^1} \leqslant C_0 h \, |u|_{H^2},$$

where

$$|u|_{H^2}^2 \equiv \sum_{i,j=1}^{2} \left\| \frac{\partial^2 u}{\partial x_i \partial x_j} \right\|_{L^2}^2.$$

Here, C_0 is a positive, numerically determined constant which is independent of h.

The following lemma is well known [4].

Lemma 1. *For any ψ in $L^2(\Omega)$, there exists a unique solution $\phi \in H^2(\Omega) \cap H_0^1(\Omega)$ of the following Poisson equation*:

$$-\Delta\phi = \psi \quad \text{in } \Omega, \qquad \phi = 0 \quad \text{on } \partial\Omega. \tag{2.1}$$

Furthermore, under Assumption 1 (*e.g.*, [2,10]), *we have*

$$\|\phi - P_{h0}\phi\|_{H_0^1} \leqslant C_0 h \, \|\psi\|_{L^2}. \tag{2.2}$$

In the following, we assume that an operator *f* satisfies the following assumptions:

A1. $f : H_0^1(\Omega) \longrightarrow L^2(\Omega)$ is continuous and maps bounded sets into bounded sets.
A2. *f* is twice Fréchet differentiable on $H_0^1(\Omega)$.

We consider the nonlinear elliptic boundary value problems of the form

$$-\Delta u = f(u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{2.3}$$

and its linearized eigenvalue problem

$$-\Delta v - f'(u)v = \lambda v \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega, \tag{2.4}$$

where u is an exact solution of the above nonlinear PDE.

Adding a normalization condition for the eigenfunction, we then consider the following system of nonlinear equations in weak form:

Find $(u, v, \lambda) \in H_0^1(\Omega) \times H_0^1(\Omega) \times \mathbb{R}$ s.t.

$$\begin{aligned} &(\nabla u, \nabla\phi)_{L^2} = (f(u), \phi)_{L^2} \quad \forall\phi \in H_0^1(\Omega), \\ &(\nabla v, \nabla\phi)_{L^2} = (\lambda v + f'(u)v, \phi)_{L^2} \quad \forall\phi \in H_0^1(\Omega), \\ &\|v\|_{L^2}^2 = 1. \end{aligned} \tag{2.5}$$

We define the inner product $\langle\cdot,\cdot\rangle_X$ and norm $\|\cdot\|_X$ in X by

$$\langle w_1, w_2\rangle_X \equiv (\nabla u_1, \nabla u_2)_{L^2} + (\nabla v_1, \nabla v_2)_{L^2} + \lambda_1\lambda_2$$
$$\text{for } w_i = (u_i, v_i, \lambda_i) \in X \ (i = 1, 2),$$

$$\|w\|_X \equiv \left(\|u\|^2_{H_0^1} + \|v\|^2_{H_0^1} + |\lambda|^2\right)^{1/2} \quad \text{for } w = (u, v, \lambda) \in X.$$

Moreover, let I_0 and I be the identity map on $H_0^1(\Omega)$ and X, respectively. We then define the projection $P_h : X \longrightarrow X_h$ by

$$P_h(u, v, \lambda) \equiv (P_{h0}u, P_{h0}v, \lambda).$$

Now, let $w_h = (u_h, \widehat{v}_h, \widehat{\lambda}_h) \in X_h$ be a finite element solution of (2.5); i.e., for all $\phi_h \in S_h$,

$$\begin{aligned} &(\nabla u_h, \nabla\phi_h)_{L^2} = (f(u_h), \phi_h)_{L^2},\\ &(\nabla\widehat{v}_h, \nabla\phi_h)_{L^2} = (\widehat{\lambda}_h\widehat{v}_h + f'(u_h)\widehat{v}_h, \phi_h)_{L^2},\\ &\|\widehat{v}_h\|^2_{L^2} = 1. \end{aligned} \tag{2.6}$$

Note that we need a verified solution of problem (2.6). Actually we have enclosed it using the interval Newton method (cf. Section 5). We will verify the existence of triple (u, v, λ) for (2.5) in the neighborhood of $(\bar{u}, \bar{v}, \widehat{\lambda}_h)$ satisfying

$$-\Delta\bar{u} = f(u_h) \quad \text{in } \Omega, \qquad \bar{u} = 0 \quad \text{on } \partial\Omega \tag{2.7}$$

and

$$-\Delta\bar{v} = \widehat{\lambda}_h\widehat{v}_h + f'(u_h)\widehat{v}_h \quad \text{in } \Omega, \qquad \bar{v} = 0 \quad \text{on } \partial\Omega. \tag{2.8}$$

These $\bar{u}$ and $\bar{v}$ are unique in $H^2(\Omega) \cap H_0^1(\Omega)$ by a well-known result for Poisson's equation. Note that

$$w_h = P_h\left(\bar{u}, \bar{v}, \widehat{\lambda}_h\right).$$

Defining $v_{01} \equiv \bar{u} - u_h,\ v_{02} \equiv \bar{v} - \widehat{v}_h$, we find that $v_{01} = \bar{u} - P_{h0}\bar{u} \in S_h^{\perp}$ and $v_{02} = \bar{v} - P_{h0}\bar{v} \in S_h^{\perp}$, where $S_h^{\perp}$ represents the orthogonal complement of S_h in $H_0^1(\Omega)$, and we can write:

$$\bar{u} = u_h + v_{01}, \qquad u_h \in S_h, \quad v_{01} \in S_h^{\perp},$$

$$\bar{v} = \widehat{v}_h + v_{02}, \qquad \widehat{v}_h \in S_h, \quad v_{02} \in S_h^{\perp}.$$

It is known that the a posteriori error estimates for finite element approximations are, in general, much better than the a priori estimates, provided that the higher order base functions in S_h are utilized (see [25] for details). Therefore, we make effective use of a posteriori estimates for v_{01} and v_{02} as in the following.

In the following, let S_h be a usual C^0-finite element subspace of $H_0^1(\Omega)$ on some appropriate triangulations or rectangulations of Ω. Let $S_h^* \subset H^1(\Omega)$ be a finite element subspace whose basis consists of the union of the basis on S_h and the base functions having nonzero values on the boundary $\partial\Omega$. Define $\bar{\nabla} u_h \in S_h^* \times S_h^*$, a vector function in two dimensions, by the L^2-projection of $\nabla u_h \in L^2(\Omega) \times L^2(\Omega)$ to $S_h^* \times S_h^*$. Then, define $\bar{\Delta} u_h \in L^2(\Omega)$ by $\bar{\Delta} u_h \equiv \nabla \cdot \bar{\nabla} u_h$. We then obtain the following estimations (cf. [25]):

$$\|v_{01}\|_{H_0^1} \leqslant \|\bar{\nabla} u_h - \nabla u_h\|_{L^2} + C_0 h \, \|\bar{\Delta} u_h + f(u_h)\|_{L^2},$$

$$\|v_{02}\|_{H_0^1} \leqslant \|\bar{\nabla} \widehat{v}_h - \nabla \widehat{v}_h\|_{L^2} + C_0 h \, \|\bar{\Delta} \widehat{v}_h + \widehat{\lambda}_h \widehat{v}_h + f'(u_h)\widehat{v}_h\|_{L^2}.$$

Note that in these estimations we used the L^2-estimates of v_{01} and v_{02}:

$$\|v_{01}\|_{L^2} \leqslant C_0 h \, \|v_{01}\|_{H_0^1}, \qquad \|v_{02}\|_{L^2} \leqslant C_0 h \, \|v_{02}\|_{H_0^1},$$

which are obtained by using the well-known Aubin–Nitsche trick (e.g., [6]).

Now, by using (2.5), (2.7) and (2.8), we have:

$$\begin{aligned} &-\Delta(u - \bar{u}) = f(u) - f(u_h), \\ &-\Delta(v - \bar{v}) = \lambda v + f'(u)v - \widehat{\lambda}_h \widehat{v}_h - f'(u_h)\widehat{v}_h, \\ &\|v\|_{L^2}^2 = 1. \end{aligned} \tag{2.9}$$

In order to verify solutions (u, v, λ) of (2.5) near $(\bar{u}, \bar{v}, \widehat{\lambda}_h)$, by writing

$$u = \bar{u} + \widetilde{u}, \quad v = \bar{v} + \widetilde{v}, \quad \lambda = \widehat{\lambda}_h + \widetilde{\lambda},$$

we can rewrite (2.9) as:

$$\begin{aligned} &-\Delta\widetilde{u} = f(\widetilde{u} + u_h + v_{01}) - f(u_h), \\ &-\Delta\widetilde{v} = f'(\widetilde{u} + u_h + v_{01})(\widetilde{v} + \widehat{v}_h + v_{02}) \\ &\qquad\quad - f'(u_h)\widehat{v}_h + (\widetilde{\lambda} + \widehat{\lambda}_h)(\widetilde{v} + \widehat{v}_h + v_{02}) - \widehat{\lambda}_h \widehat{v}_h, \\ &\|\widetilde{v} + \widehat{v}_h + v_{02}\|_{L^2}^2 = 1. \end{aligned}$$

Thus, using the following compact map on *X*,

$$F(w) \equiv \begin{pmatrix} (-\Delta)^{-1}\{f(\widetilde{u} + u_h + v_{01}) - f(u_h)\}, \\ (-\Delta)^{-1}\{f'(\widetilde{u} + u_h + v_{01})(\widetilde{v} + \widehat{v}_h + v_{02}) - f'(u_h)\widehat{v}_h \\ +(\widetilde{\lambda} + \widehat{\lambda}_h)(\widetilde{v} + \widehat{v}_h + v_{02}) - \widehat{\lambda}_h \widehat{v}_h\}, \\ \widetilde{\lambda} + \|\widetilde{v} + \widehat{v}_h + v_{02}\|_{L^2}^2 - 1, \end{pmatrix} \tag{2.10}$$

where $(-\Delta)^{-1}$ is the solution operator for the Poisson equation with homogeneous Dirichlet boundary conditions, we have the fixed point equation

$$w = F(w) \quad \text{for } w = (\widetilde{u}, \widetilde{v}, \widetilde{\lambda}). \tag{2.11}$$

2.2. *Verification conditions and algorithm in a computer*

We now make the following assumption.

Assumption 2. Set $\rho \equiv (-v_{01}, -v_{02}, 0) \in X_h^\perp$, where $X_h^\perp$ stands for the orthogonal complement of X_h in X, and define $F'(\rho)$ as the Fréchet derivative of F at ρ. Assume that the restriction to X_h of the operator $P_h[I - F'(\rho)] : X \longrightarrow X_h$ has the inverse

$$[I - F'(\rho)]_h^{-1} : X_h \longrightarrow X_h.$$

The validity of this assumption can be numerically checked in actual computations.

Now, as in [14,15], we decompose (2.11) into finite- and infinite-dimensional parts:

$$P_h w = P_h F(w), \qquad (I - P_h)w = (I - P_h)F(w). \tag{2.12}$$

We use a Newton-like method only for the finite-dimensional part, represented by the first equation in (2.12). Defining the Newton-like operator

$$N_h(w) \equiv P_h w - [I - F'(\rho)]_h^{-1}(P_h w - P_h F(w)),$$

we define the operator $T : X \longrightarrow X$ as

$$T(w) \equiv N_h(w) + (I - P_h)F(w). \tag{2.13}$$

Then T becomes a compact map on X, and the equivalence relation

$$w = T(w) \Longleftrightarrow w = F(w) \tag{2.14}$$

holds.

Although the following arguments are almost the same as those given in [9], we include them with full detail to make this paper self-contained.

We can write an arbitrary element $w \in X$ as

$$w = (z_h + z_\perp, v_h + v_\perp, \mu), z_h, v_h \in S_h, \qquad z_\perp, v_\perp \in S_h^\perp, \quad \mu \in \mathbb{R}$$

with $z_h = \sum_{j=1}^M z_j\phi_j$, $v_h = \sum_{j=1}^M v_j\phi_j$, where $M = \dim S_h$, $\{\phi_j\}_{j=1}^M$ is a basis of S_h, $(z_j)_{j=1}^M$ and $(v_j)_{j=1}^M$ are real vectors. For above w we define the following notation:

$$\begin{aligned}
&(w)_i \equiv |z_i|, \quad i = 1, \ldots, M,\\
&(w)_i \equiv |v_{i-M}|, \quad i = M+1, \ldots, 2M,\\
&(w)_{2M+1} \equiv \|z_\perp\|_{H_0^1},\\
&(w)_{2M+2} \equiv \|v_\perp\|_{H_0^1},\\
&(w)_{2M+3} \equiv |\mu|.
\end{aligned}$$

We intend to find a fixed point to (2.11) in a set W, referred to as a "candidate set". Given a vector $(W_1, \ldots, W_{2M+3})^{\mathrm{t}}$ such that $W_i > 0$ $(i = 1, \ldots, 2M+3)$, we define the corresponding candidate set W by

$$W \equiv \{w \in X \mid (w)_i \leqslant W_i \ (i = 1, \ldots, 2M+3)\}. \tag{2.15}$$

Now let T' be the Fréchet derivative of T. By the method described in the following, we choose two vectors $(Y_1, \ldots, Y_{2M+3})^{\mathrm{t}}$, $Y_i > 0$ $(i = 1, \ldots, 2M + 3)$ and $(Z_1, \ldots, Z_{2M+3})^{\mathrm{t}}$, $Z_i > 0$ $(i = 1, \ldots, 2M + 3)$ such that

$$(T(0))_i \leqslant Y_i, \quad i = 1, \ldots, 2M + 3, \tag{2.16}$$

$$(T'(w_1)w_2)_i \leqslant Z_i, \qquad i = 1, \ldots, 2M + 3, \quad \text{for all } w_1, w_2 \in W. \tag{2.17}$$

We now derive the following theorem in which the verification condition is described.

Theorem 1. *If a candidate set W defined by* (2.15) *satisfies*

$$Y_i + Z_i < W_i \quad (i = 1, \ldots, 2M + 3), \tag{2.18}$$

then there exists a fixed point of T in

$$K \equiv \{v \in X \mid (v)_i \leqslant Y_i + Z_i \ (i = 1, \ldots, 2M + 3)\}. \tag{2.19}$$

Moreover, this fixed point is unique within the set W.

In order to prove this theorem, we derive two preliminary lemmas.

Defining the norm $\| \cdot \|_W$ by

$$\|x\|_W \equiv \max_{1 \leqslant i \leqslant 2M+3} \frac{(x)_i}{W_i}, \quad x \in X, \tag{2.20}$$

we have the following lemma.

Lemma 2. *For each $x \in X$,*

$$\sup_{w \in W} \|T'(w)x\|_W \leqslant \max_{1 \leqslant i \leqslant 2M+3} \frac{Z_i}{W_i} \|x\|_W.$$

Proof. Since $T'(w)$ is linear, we obtain for $x \in X$,

$$\sup_{w \in W} \|T'(w)x\|_W = \|x\|_W \sup_{w \in W} \left\| T'(w) \frac{x}{\|x\|_W} \right\|_W .$$

Then, by the definition of $\| \cdot \|_W$, we see that $x/\|x\|_W \in W$ and this implies

$$\sup_{w \in W} \left\| T'(w) \frac{x}{\|x\|_W} \right\|_W \leqslant \max_{1 \leqslant i \leqslant 2M+3} \frac{Z_i}{W_i}.$$

This proves Lemma 2. $\square$

Lemma 3. *For any $w_1, w_2 \in W$,*

$$\|T(w_1) - T(w_2)\|_W \leqslant \sup_{s \in [0,1]} \|T'(sw_1 + (1 - s)w_2)(w_1 - w_2)\|_W. \tag{2.21}$$

Proof. Defining

$$g(s) \equiv T(sw_1 + (1-s)w_2),$$

apply the mean value theorem to obtain the desired conclusion. $\square$

With Lemmas 2 and 3, we can now prove Theorem 1. As usual, the image $J(K)$ of an arbitrary operator J and arbitrary set K is defined by

$$J(K) \equiv \{J(k) \mid k \in K\}.$$

Proof of Theorem 1. We first show the inclusion ralation $T(W) \subset W$. By the similar proof of Lemma 3, for $i = 1, \ldots, 2M+3$, we have

$$(T(w) - T(0))_i \leqslant \sup_{s\in[0,1]} (T'(sw)w)_i \quad \text{for all } w \in W.$$

Therefore, by (2.17), for $i = 1, \ldots, 2M+3$,

$$(T(w) - T(0))_i \leqslant Z_i$$

holds. Hence, we have

$$\begin{aligned}(T(w))_i &\leqslant (T(0))_i + (T(w) - T(0))_i \\ &\leqslant Y_i + Z_i \\ &< W_i,\end{aligned}$$

from which we obtain $T(w) \in W$. This implies $T(W) \subset W$.

We next prove that, for some $0 < k < 1$ and for all $w_1, w_2 \in W$,

$$\|T(w_2) - T(w_1)\|_W \leqslant k\|w_2 - w_1\|_W.$$

Since W is convex, by Lemmas 2 and 3, we have

$$\begin{aligned}\|T(w_2) - T(w_1)\|_W &\leqslant \sup_{s\in[0,1]} \|T'(sw_2 + (1-s)w_1)(w_2 - w_1)\|_W \\ &\leqslant \sup_{w_3\in W} \|T'(w_3)(w_2 - w_1)\|_W \\ &\leqslant \max_{1\leqslant i\leqslant 2M+3} \frac{Z_i}{W_i} \|w_2 - w_1\|_W.\end{aligned}$$

Thus, $Y_i > 0$ $(i = 1, \ldots, 2M+3)$ and (2.18) imply that there exists a positive real value k satisfying

$$\frac{Z_i}{W_i} < \frac{Z_i + Y_i}{W_i} \leqslant k < 1 \quad (i = 1, \ldots, 2M+3).$$

Therefore, applying Banach's fixed point theorem to T, Theorem 1 is proved. $\square$

In the following, we describe the procedure to choose positive vectors $(Y_1, \ldots, Y_{2M+3})^{\mathrm{t}}$ and $(Z_1, \ldots, Z_{2M+3})^{\mathrm{t}}$, satisfying (2.16) and (2.17), respectively.

As usual, we define the absolute value of any interval A as $|A| \equiv \max_{a\in A} |a|$.

Since

$$\begin{aligned}T(0) &= N_h(0) + (I - P_h)F(0)\\ &= -[I - F'(\rho)]_h^{-1}(-P_hF(0)) + (I - P_h)F(0)\\ &= [I - F'(\rho)]_h^{-1}P_hF(0) + (I - P_h)F(0)\end{aligned}$$

holds, for $Y_1, \ldots, Y_{2M}$ and Y_{2M+3}, we first calculate the interval vector $(\widetilde{Y}_1, \ldots, \widetilde{Y}_{2M}, \widetilde{Y}_{2M+3})^{\mathrm{t}}$ satisfying

$$\begin{aligned}\{P_hT(0)\} &= \{[I - F'(\rho)]_h^{-1}P_hF(0)\}\\ &\subset \left(\sum_{j=1}^{M}\widetilde{Y}_j\phi_j, \sum_{j=1}^{M}\widetilde{Y}_{j+M}\phi_j, \widetilde{Y}_{2M+3}\right).\end{aligned} \tag{2.22}$$

It is then sufficient to set

$$Y_i = |\widetilde{Y}_i| \quad (i = 1, \ldots, 2M, 2M+3). \tag{2.23}$$

To calculate the interval vector $(\widetilde{Y}_1, \ldots, \widetilde{Y}_{2M}, \widetilde{Y}_{2M+3})^{\mathrm{t}}$ satisfying (2.22), we rewrite the single-point set $\mathscr{Y} = \{[I - F'(\rho)]_h^{-1}P_hF(0)\} = \{P_hT(0)\} \subset X_h$ as

$$\begin{aligned}\mathscr{Y} \equiv \{y \in X_h \mid &\text{ for all } i = 1, \ldots, 2M+1\\ &\langle [I - F'(\rho)]_h y,\ \Phi_i\rangle_X = \langle P_hF(0),\ \Phi_i\rangle_X\},\end{aligned} \tag{2.24}$$

where we have used the basis $\Phi_1, \ldots, \Phi_{2M+1}$ of X_h given by $\Phi_i \equiv (\phi_i, 0, 0)$ $(i = 1, \ldots, M)$, $\Phi_i \equiv (0, \phi_{i-M}, 0)$ $(i = M+1, \ldots, 2M)$, $\Phi_{2M+1} \equiv (0, 0, 1)$.

In the actual computation, as shown in the following, we can obtain the interval hull of $\mathscr{Y}$ (denoted by $\boxed{\mathscr{Y}}$) by solving the linear system of equations in (2.24). Then we can determine $(\widetilde{Y}_1, \ldots, \widetilde{Y}_{2M}, \widetilde{Y}_{2M+3})^{\mathrm{t}}$ as

$$\left(\sum_{j=1}^{M}\widetilde{Y}_j\phi_j,\ \sum_{j=1}^{M}\widetilde{Y}_{j+M}\phi_j,\ \widetilde{Y}_{2M+3}\right) \equiv \boxed{\mathscr{Y}}. \tag{2.25}$$

Observe that for Φ_i $(1 \leqslant i \leqslant M)$ and $y \equiv (\sum_{j=1}^{M} y_j\phi_j,\ \sum_{j=1}^{M} y_{j+M}\phi_j,\ y_{2M+3})$, we have

$$\langle [I - F'(\rho)]_h y, \Phi_i\rangle_X = \sum_{j=1}^{M} y_j\{(\nabla\phi_j, \nabla\phi_i)_{L^2} - (f'(u_h)\phi_j, \phi_i)_{L^2}\}, \tag{2.26}$$

and for Φ_i $(M+1 \leqslant i \leqslant 2M)$, we have

$$\langle [I - F'(\rho)]_h y, \Phi_i\rangle_X = \sum_{j=1}^{M} y_{j+M}(\nabla\phi_j, \nabla\phi_{i-M})_{L^2}$$

$$
\begin{aligned}
&-\sum_{j=1}^{M} y_j (f''(u_h)\widehat{v}_h\phi_j, \phi_{i-M})_{L^2} \\
&-\sum_{j=1}^{M} y_{j+M}(\widehat{\lambda}_h\phi_j + f'(u_h)\phi_j, \phi_{i-M})_{L^2} \\
&-y_{2M+3}(\widehat{v}_h, \phi_{i-M})_{L^2},
\end{aligned}
\tag{2.27}
$$

and for Φ_{2M+1}, we obtain

$$
\langle [I - F'(\rho)]_h y, \Phi_{2M+1}\rangle_X = -2\left(\widehat{v}_h, \sum_{j=1}^{M} y_{j+M}\phi_j\right)_{L^2}. \tag{2.28}
$$

Moreover, for Φ_i $(1 \leqslant i \leqslant M)$, we have

$$
\langle P_h F(0), \Phi_i\rangle_X = (f(u_h + v_{01}) - f(u_h), \phi_i)_{L^2}, \tag{2.29}
$$

and for Φ_i $(M+1 \leqslant i \leqslant 2M)$, we have

$$
\begin{aligned}
&\langle P_h F(0), \Phi_i\rangle_X \\
&\quad = (f'(u_h + v_{01})(\widehat{v}_h + v_{02}) - f'(u_h)\widehat{v}_h + \widehat{\lambda}_h v_{02}, \phi_{i-M})_{L^2},
\end{aligned}
\tag{2.30}
$$

and for Φ_{2M+1}, we obtain

$$
\langle P_h F(0), \Phi_{2M+1}\rangle_X = \int_\Omega \left(v_{02}^2 + 2\widehat{v}_h v_{02}\right)\,\mathrm{d}x. \tag{2.31}
$$

Now, in order to obtain the set $\mathscr{Y}$, we define the $(2M+1)\times(2M+1)$ matrix $G \equiv (g_{ij})_{1\leqslant i,j\leqslant 2M+1}$ by

$$
\begin{aligned}
g_{ij} &= (\nabla\phi_i, \nabla\phi_j)_{L^2} + (-f'(u_h)\phi_i, \phi_j)_{L^2} \quad (1 \leqslant i,\ j \leqslant M),\\
g_{ij} &= (-f''(u_h)\widehat{v}_h\phi_{i-M}, \phi_j)_{L^2} \quad (M+1 \leqslant i \leqslant 2M,\ 1 \leqslant j \leqslant M),\\
g_{ij} &= 0 \quad (1 \leqslant i \leqslant M,\ M+1 \leqslant j \leqslant 2M),\\
g_{ij} &= (\nabla\phi_{i-M}, \nabla\phi_{j-M})_{L^2} + (-f'(u_h)\phi_{i-M}, \phi_{j-M})_{L^2}\\
&\qquad -\widehat{\lambda}_h(\phi_{i-M}, \phi_{j-M})_{L^2} \quad (M+1 \leqslant i,\ j \leqslant 2M),\\
g_{i,2M+1} &= 0 \quad (1 \leqslant i \leqslant M),\\
g_{i,2M+1} &= -(\widehat{v}_h, \phi_{i-M})_{L^2} \quad (M+1 \leqslant i \leqslant 2M),\\
g_{2M+1,j} &= 0 \quad (1 \leqslant j \leqslant M),\\
g_{2M+1,j} &= -2(\widehat{v}_h, \phi_{j-M})_{L^2} \quad (M+1 \leqslant j \leqslant 2M),\\
g_{2M+1,2M+1} &= 0,
\end{aligned}
\tag{2.32}
$$

and the interval vector $\mathbf{r} \equiv ([-r_i, r_i])_{i=1}^{2M+1}$ by

$$
\begin{aligned}
r_i &\equiv |(f(u_h + v_{01}) - f(u_h), \phi_i)_{L^2}| \quad (i = 1, \ldots, M),\\
r_i &\equiv |(f'(u_h + v_{01})(\widehat{v}_h + v_{02}) - f'(u_h)\widehat{v}_h + \widehat{\lambda}_h v_{02}, \phi_{i-M})_{L^2}|
\end{aligned}
$$

$$(i = M+1, \ldots, 2M),$$
$$r_{2M+1} \equiv \left| \int_\Omega \left(v_{02}^2 + 2\widehat{v}_h v_{02} \right) \mathrm{d}x \right|.$$

In order to evaluate this interval vector $\mathbf{r}$, we used a posteriori estimates for v_{01} and v_{02} which are derived in Section 2.1.

If we can numerically check the validity of Assumption 2 in actual computation, then the interval vector $(\widetilde{Y}_1, \ldots, \widetilde{Y}_{2M}, \widetilde{Y}_{2M+3})^{\mathrm{t}}$ in (2.25) is determined by

$$(\widetilde{Y}_1, \ldots, \widetilde{Y}_{2M}, \widetilde{Y}_{2M+3})^{\mathrm{t}} \equiv G^{-1}\mathbf{r}. \tag{2.33}$$

We can enclose $G^{-1}\mathbf{r}$ using some interval library, e.g., PROFIL (cf. [5]).

Furthermore, Y_{2M+1} and Y_{2M+2} can be set as

$$Y_{2M+1} = C_0 h \, \| f(u_h + v_{01}) - f(u_h) \|_{L^2},$$

$$Y_{2M+2} = C_0 h \, \| f'(u_h + v_{01})(\widehat{v}_h + v_{02}) - f'(u_h)\widehat{v}_h + \widehat{\lambda}_h v_{02} \|_{L^2}$$

by the use of Lemma 1.

Next, we choose a vector $(Z_1, \ldots, Z_{2M+3})^{\mathrm{t}}$ satisfying (2.17). Since

$$\begin{aligned} T'(w_1)w_2 &= N_h'(w_1)w_2 + (I - P_h)F'(w_1)w_2 \\ &= [I - F'(\rho)]_h^{-1} P_h (F'(w_1)w_2 - F'(\rho)P_h w_2) + (I - P_h)F'(w_1)w_2 \end{aligned}$$

holds, for $Z_1, \ldots, Z_{2M}, Z_{2M+3}$, we first determine the interval vector $(\widetilde{Z}_1, \ldots, \widetilde{Z}_{2M}, \widetilde{Z}_{2M+3})^{\mathrm{t}}$ satisfying for all $w_1, w_2 \in W$,

$$\begin{aligned} \{P_h T'(w_1)w_2\} &= \left\{ [I - F'(\rho)]_h^{-1} P_h (F'(w_1)w_2 - F'(\rho)P_h w_2) \right\} \\ &\subset \left(\sum_{j=1}^{M} \widetilde{Z}_j \phi_j, \ \sum_{j=1}^{M} \widetilde{Z}_{j+M} \phi_j, \ \widetilde{Z}_{2M+3} \right), \end{aligned} \tag{2.34}$$

and then set

$$Z_i = |\widetilde{Z}_i| \quad (i = 1, \ldots, 2M, 2M+3). \tag{2.35}$$

To calculate the interval vector $(\widetilde{Z}_1, \ldots, \widetilde{Z}_{2M}, \widetilde{Z}_{2M+3})^{\mathrm{t}}$ satisfying (2.34), we rewrite the set $\mathscr{Z} = \{[I - F'(\rho)]_h^{-1} P_h (F'(w_1)w_2 - F'(\rho)P_h w_2) \mid w_1, w_2 \in W\} = \{P_h T'(w_1)w_2 \mid w_1, w_2 \in W\} \subset X_h$ as

$$\begin{aligned} \mathscr{Z} \equiv \Big\{ z \in X_h \mid & \text{there exist } w_1, w_2 \in W \text{ such that, for all } i = 1, \ldots, 2M+1, \\ & \langle [I - F'(\rho)]_h z, \Phi_i \rangle_X = \langle P_h (F'(w_1)w_2 - F'(\rho)P_h w_2), \Phi_i \rangle_X \Big\}. \end{aligned} \tag{2.36}$$

In analogy to our treatment of $\mathscr{Y}$, we can obtain the interval hull of $\mathscr{Z}$ (denoted by $\boxed{\mathscr{Z}}$) by solving the linear system of equations using the interval right-hand side, as we now do.

Observe that for Φ_i $(1 \leqslant i \leqslant M)$ and for all $w_1, w_2 \in W$, we have

$$\begin{aligned}&\langle P_h(F'(w_1)w_2 - F'(\rho)P_h w_2), \Phi_i\rangle_X\\&\quad = (f'(u_1+u_h+v_{01})u_2 - f'(u_h)(P_{h0}u_2), \phi_i)_{L^2},\end{aligned} \tag{2.37}$$

and for Φ_i $(M+1 \leqslant i \leqslant 2M)$, we obtain

$$\begin{aligned}&\langle P_h(F'(w_1)w_2 - F'(\rho)P_h w_2), \Phi_i\rangle_X\\&\quad = \big(f''(u_1+u_h+v_{01})(v_1+\widehat{v}_h+v_{02})u_2 + f'(u_1+u_h+v_{01})v_2\\&\qquad +(\lambda_1+\widehat{\lambda}_h)v_2 + (v_1+\widehat{v}_h+v_{02})\lambda_2 - f''(u_h)\widehat{v}_h P_{h0}u_2\\&\qquad -f'(u_h)P_{h0}v_2 - \widehat{\lambda}_h P_{h0}v_2 - \lambda_2\widehat{v}_h, \phi_{i-M}\big)_{L^2},\end{aligned} \tag{2.38}$$

and for Φ_{2M+1}, we have

$$\begin{aligned}&\langle P_h(F'(w_1)w_2 - F'(\rho)P_h w_2), \Phi_{2M+1}\rangle_X\\&\quad = 2\int_\Omega (v_1+\widehat{v}_h+v_{02})v_2 \,\mathrm{d}x - 2\int_\Omega \widehat{v}_h(P_{h0}v_2)\,\mathrm{d}x\end{aligned} \tag{2.39}$$

for $w_i = (u_i, v_i, \lambda_i), u_i, v_i \in H_0^1(\Omega),\ \lambda_i \in \mathbb{R}$ $(i = 1, 2)$.

Therefore, in order to obtain the set $\boxed{\mathscr{Z}}$, we use the matrix G determined by (2.32) and the interval vector $\mathbf{r} \equiv ([-r_i, r_i])_{i=1}^{2M+1}$ for which

$$\begin{aligned}r_i \equiv &\sup_{w_j \in W (j=1,2)} \left|\big(f'(u_1+u_h+v_{01})u_2 - f'(u_h)(P_{h0}u_2), \phi_i\big)_{L^2}\right|\\&(i = 1, \ldots, M),\end{aligned}$$

$$\begin{aligned}r_i \equiv \sup_{w_j \in W (j=1,2)} &\big|\big(f''(u_1+u_h+v_{01})(v_1+\widehat{v}_h+v_{02})u_2\\&+f'(u_1+u_h+v_{01})v_2 + (\lambda_1+\widehat{\lambda}_h)v_2 + (v_1+\widehat{v}_h+v_{02})\lambda_2\\&-f''(u_h)\widehat{v}_h P_{h0}u_2 - f'(u_h)P_{h0}v_2 - \widehat{\lambda}_h P_{h0}v_2\\&-\lambda_2\widehat{v}_h, \phi_{i-M})_{L^2}\big| \quad (i = M+1, \ldots, 2M),\end{aligned}$$

$$r_{2M+1} \equiv \sup_{w_j \in W (j=1,2)} \left|2\int_\Omega (v_1+\widehat{v}_h+v_{02})v_2\,\mathrm{d}x - 2\int_\Omega \widehat{v}_h(P_{h0}v_2)\,\mathrm{d}x\right|.$$

Then we set

$$\big(\widetilde{Z}_1, \ldots, \widetilde{Z}_{2M}, \widetilde{Z}_{2M+3}\big)^{\mathrm{t}} = G^{-1}\mathbf{r}. \tag{2.40}$$

We can also estimate Z_{2M+1} and Z_{2M+2} by

$$Z_{2M+1} = \sup_{w_j \in W (j=1,2)} C_0 h\, \|f'(u_1+u_h+v_{01})u_2\|_{L^2},$$

$$Z_{2M+2} = \sup_{w_j \in W (j=1,2)} C_0 h\, \|f''(u_1+u_h+v_{01})(v_1+\widehat{v}_h+v_{02})u_2$$

$$+ f'(u_1 + u_h + v_{01})v_2 + (\lambda_1 + \widehat{\lambda}_h)v_2 + (v_1 + \widehat{v}_h + v_{02})\lambda_2\|_{L^2}$$

in similar way as Y_{2M+1} and Y_{2M+2}.

Now, we describe an algorithm for finding a vector $(W_1, \dots, W_{2M+3})^t$ which satisfies the verification condition (2.18). Since $(Z_i)_{i=1}^{2M+3}$ depends on W, we write Z_i as $Z_i(W)$. We use the following iteration method.

Algorithm.

1. Find a positive vector $(Y_1, \dots, Y_{2M+3})^t$ satisfying condition (2.16).
2. Set $W_i \leftarrow Y_i$ $(i = 1, \dots, 2M + 3)$.
3. Find a positive vector $(Z_1(W), \dots, Z_{2M+3}(W))^t$ satisfying (2.17).
4. Check the verification condition in Theorem 1;
   $Y_i + Z_i(W) < W_i$ $(i = 1, \dots, 2M + 3)$.
   If the condition is satisfied, then the verification has succeeded.
   If not, set
   $$W_i \leftarrow (1 + \delta)(Y_i + Z_i) \quad (i = 1, \dots, 2M + 3), \tag{2.41}$$
   where δ $(0 < \delta \ll 1)$ represents an inflation parameter (cf. [20,24]), increase the iteration number by 1, and return to step 3.
5. If the maximum iteration number is exceeded without (2.18) being satisfied, the verification has failed.

Now we assume that there exists a candidate set W satisfying the hypothesis in Theorem 1. We then define

$$\begin{aligned}
U_i &\equiv W_i \ (i = 1, \dots, M), \quad U_{M+1} \equiv W_{2M+1}, \\
V_i &\equiv W_{M+i} \ (i = 1, \dots, M), \quad V_{M+1} \equiv W_{2M+2}, \\
\Lambda_0 &\equiv W_{2M+3},
\end{aligned}$$

and set

$$U \equiv \{u \in H_0^1(\Omega) \mid (u)_i \leqslant U_i \ (i = 1, \dots, M + 1)\}, \tag{2.42}$$

$$V \equiv \{v \in H_0^1(\Omega) \mid (v)_i \leqslant V_i \ (i = 1, \dots, M + 1)\}, \tag{2.43}$$

$$\Lambda \equiv \{\lambda \in \mathbb{R} \mid |\lambda| \leqslant \Lambda_0\}, \tag{2.44}$$

where

$$\begin{aligned}
(u)_i &\equiv |u_i|, \quad i = 1, \dots, M, \\
(u)_{M+1} &\equiv \|u_\perp\|_{H_0^1}
\end{aligned}$$

with

$$u = \sum_{j=1}^{M} u_j \phi_j + u_\perp, \qquad u_1, \ldots, u_M \in \mathbb{R}, \quad u_\perp \in S_h^\perp.$$

Then we have $W = U \times V \times \Lambda$.

By Theorem 1, the local uniqueness of a *triple* is confirmed in $U \times V \times \Lambda$. But this does not imply directly that the eigenvalue is unique in Λ, because there may exist another eigenvalue in Λ corresponding to another eigenfunction ontside V or solution for original PDE outside U.

We therefore must show the local uniqueness individually for each solution of nonlinear PDE, eigenfunction and eigenvalue in U, V and Λ, respectively.

3. The local uniqueness of solutions

Let U, V, Λ and W be the sets defined at the end of Section 2, i.e., we assume that condition (2.18) holds. Our aim in this section is to prove the uniqueness of a solution of nonlinear PDE in U, of an eigenvalue in Λ and of an eigenfunction in V separately.

In the following, we use the similar kind of proof technique to that in [9]. We denote the operator $T : X \longrightarrow X$ defined by (2.13) as

$$T(u, v, \lambda) = \begin{pmatrix} T_1(u, v, \lambda) \\ T_2(u, v, \lambda) \\ T_3(u, v, \lambda) \end{pmatrix}, \tag{3.1}$$

where $T_i,\ 1 \leqslant i \leqslant 3$, are operators such that

$$T_1, T_2 : X \longrightarrow H_0^1(\Omega), \tag{3.2}$$

$$T_3 : X \longrightarrow \mathbb{R}. \tag{3.3}$$

Now, for $z \in H_0^1(\Omega)$, we write

$$z = \sum_{j=1}^{M} z_j \phi_j + z_\perp,$$

where $z_1, \ldots, z_M \in \mathbb{R},\ z_\perp \in S_h^\perp$, and define the norm $\|\cdot\|_U$ and $\|\cdot\|_V$ by

$$\begin{aligned} \|z\|_U &\equiv \max\left\{\max_{j=1,\ldots,M} \frac{|z_j|}{U_j}, \frac{\|z_\perp\|_{H_0^1}}{U_{M+1}}\right\}, \\ \|z\|_V &\equiv \max\left\{\max_{j=1,\ldots,M} \frac{|z_j|}{V_j}, \frac{\|z_\perp\|_{H_0^1}}{V_{M+1}}\right\}. \end{aligned} \tag{3.4}$$

For fixed $u \in U,\ \lambda \in \Lambda$, we define the operator $p_{u,\lambda} : H_0^1(\Omega) \longrightarrow H_0^1(\Omega)$ by

$$p_{u,\lambda}(v) \equiv T_2(u, v, \lambda). \tag{3.5}$$

If (2.18) holds, then we have

$$p_{u,\lambda}(v) \in \mathrm{int}(V) \quad \text{for all } v \in V. \tag{3.6}$$

We then have the following lemma.

Lemma 4. *There exists a fixed point of* $p_{u,\lambda}$ *in V for each* $u \in U$ *and* $\lambda \in \Lambda$, *and this fixed point is unique in V. Moreover, when we denote this fixed point as* $v_{u,\lambda}$, *the equality*

$$\int_\Omega (v_{u,\lambda} + \bar{v})^2 \, \mathrm{d}x = 1 \tag{3.7}$$

holds. (*Recall that* $\bar{v} = \widehat{v}_h + v_{02}$).

Proof. From the proof of Theorem 1, for some $0 < k < 1$,

$$\|T(w_2) - T(w_1)\|_W \leqslant k\|w_2 - w_1\|_W, \qquad 0 < k < 1, \quad w_1, w_2 \in W$$

holds. This implies that, for any $w_1 = (u, v_1, \lambda),\ w_2 = (u, v_2, \lambda)$ in W,

$$\|T(u, v_2, \lambda) - T(u, v_1, \lambda)\|_W \leqslant k\|(u, v_2, \lambda) - (u, v_1, \lambda)\|_W.$$

By definition, we have

$$\begin{aligned}
&\|T(u, v_2, \lambda) - T(u, v_1, \lambda)\|_W \\
&\quad = \left\| \begin{pmatrix} T_1(u, v_2, \lambda) \\ T_2(u, v_2, \lambda) \\ T_3(u, v_2, \lambda) \end{pmatrix} - \begin{pmatrix} T_1(u, v_1, \lambda) \\ T_2(u, v_1, \lambda) \\ T_3(u, v_1, \lambda) \end{pmatrix} \right\|_W \\
&\quad = \left\| \begin{pmatrix} T_1(u, v_2, \lambda) \\ p_{u,\lambda}(v_2) \\ T_3(u, v_2, \lambda) \end{pmatrix} - \begin{pmatrix} T_1(u, v_1, \lambda) \\ p_{u,\lambda}(v_1) \\ T_3(u, v_1, \lambda) \end{pmatrix} \right\|_W \\
&\quad = \max\Bigg\{ \|T_1(u, v_2, \lambda) - T_1(u, v_1, \lambda)\|_U,\ \|p_{u,\lambda}(v_2) - p_{u,\lambda}(v_1)\|_V, \\
&\qquad\qquad \frac{|T_3(u, v_2, \lambda) - T_3(u, v_1, \lambda)|}{\Lambda_0} \Bigg\},
\end{aligned}$$

and

$$\|(u, v_2, \lambda) - (u, v_1, \lambda)\|_W = \|(0, v_2 - v_1, 0)\|_W = \|v_2 - v_1\|_V.$$

Hence, for all $v_1, v_2 \in V$ and for the above k, we have

$$\|p_{u,\lambda}(v_2) - p_{u,\lambda}(v_1)\|_V \leqslant k\|v_2 - v_1\|_V. \tag{3.8}$$

By (3.6) and (3.8), applying Banach's fixed point theorem for $p_{u,\lambda}$, the first part of Lemma 4 is proved.

When we denote the above fixed point as $v_{u,\lambda}$, we show

$$\int_\Omega (v_{u,\lambda} + \bar{v})^2 \, \mathrm{d}x = 1.$$

Noting the role of the finite-dimensional part of the operator T, we see that

$$\begin{pmatrix} P_{h0}T_1(u,v,\lambda) \\ P_{h0}T_2(u,v,\lambda) \\ T_3(u,v,\lambda) \end{pmatrix} = \begin{pmatrix} P_{h0}u \\ P_{h0}v \\ \lambda \end{pmatrix} - [I - F'(\rho)]_h^{-1}\{P_h(u,v,\lambda) - P_hF(u,v,\lambda)\}$$

for $(u,v,\lambda) \in X$. Then, using the definition of $p_{u,\lambda}$ we represent the above equality as

$$P_h(u,v,\lambda) - P_hF(u,v,\lambda) = [I - F'(\rho)]_h \begin{pmatrix} P_{h0}(u) - P_{h0}T_1(u,v,\lambda) \\ P_{h0}(v) - P_{h0}p_{u,\lambda}(v) \\ \lambda - T_3(u,v,\lambda) \end{pmatrix}.$$

Comparing the third components on each side of this equality, we have

$$1 - \int_\Omega (v+\bar{v})^2\,\mathrm{d}x = -2\int_\Omega \widehat{v}_h P_{h0}(v - p_{u,\lambda}(v))\,\mathrm{d}x.$$

Since in the case $v = v_{u,\lambda}$, we have $v_{u,\lambda} - p_{u,\lambda}(v_{u,\lambda}) = 0$, the second part of Lemma 4 follows. □

Now, we obtain the following lemma, which derives the local uniqueness of eigenvalues.

Lemma 5. *Assume that* (2.18) *in Theorem* 1 *is satisfied and let* $(u^* - \bar{u}, v^* - \bar{v}, \lambda^* - \widehat{\lambda}_h)$ *be a fixed point of T. If* $u^* - \bar{u} \in U$, $\lambda^* - \widehat{\lambda}_h \in \Lambda$, *then either* $v^* - \bar{v} \in V$ *or* $-v^* - \bar{v} \in V$ *holds.*

Proof. Since there exists a unique fixed point of $p_{u^*-\bar{u},\lambda^*-\widehat{\lambda}_h}$ in V by Lemma 4, we denote it as $\check{v}$ and define $\check{v}^* \equiv \check{v} + \bar{v}$.

We now assume that $v^* \neq \pm\check{v}^*$.

From the fact that

$$\int_\Omega (\check{v}^*)^2\,\mathrm{d}x = \int_\Omega (\check{v} + \bar{v})^2\,\mathrm{d}x = 1$$

by Lemma 4, defining

$$\kappa \equiv \int_\Omega v^*\check{v}^*\,\mathrm{d}x,$$

we obtain

$$|\kappa| = \left|\int_\Omega v^*\check{v}^*\,\mathrm{d}x\right| \leqslant \|v^*\|_{L^2}\,\|\check{v}^*\|_{L^2} = 1$$

by Schwarz' inequality. Noting that the equality here holds only in the case $v^* = \pm\check{v}^*$ and using our assumption $v^* \neq \pm\check{v}^*$ we have $|\kappa| \neq 1$.

Now, we make use of a kind of homotopy technique in the following (cf. [9]). For each $t \in \mathbb{R}$, we define

$$g(t) \equiv \xi(t)v^* + \eta(t)\check{v}^*, \tag{3.9}$$

where the functions $\xi(t)$ and $\eta(t)$ are defined by

$$\xi(t) \equiv \frac{1}{\sqrt{2}}\left(\frac{\cos t}{\sqrt{1+\kappa}} + \frac{\sin t}{\sqrt{1-\kappa}}\right), \qquad \eta(t) \equiv \frac{1}{\sqrt{2}}\left(\frac{\cos t}{\sqrt{1+\kappa}} - \frac{\sin t}{\sqrt{1-\kappa}}\right).$$

Then we obtain $\|g(t)\|_{L^2} = 1$ by a straightforward calculation. We can find that for all $t \in \mathbb{R}$,

$$\begin{aligned}
&T(u^* - \bar{u},\ g(t) - \bar{v},\ \lambda^* - \widehat{\lambda}_h) \\
&= \begin{pmatrix} P_{h0}(u^* - \bar{u}) \\ P_{h0}(\xi(t)v^* + \eta(t)\check{v}^* - \bar{v}) \\ \lambda^* - \widehat{\lambda}_h \end{pmatrix} \\
&\quad -[I - F'(\rho)]_h^{-1} \left\{ \begin{pmatrix} P_{h0}(u^* - \bar{u}) \\ P_{h0}(\xi(t)v^* + \eta(t)\check{v}^* - \bar{v}) \\ \lambda^* - \widehat{\lambda}_h \end{pmatrix} \right. \\
&\quad - \left. \begin{pmatrix} P_{h0}(-\Delta)^{-1}\{f(u^* - \bar{u} + u_h + v_{01}) - f(u_h)\} \\ P_{h0}(-\Delta)^{-1}\{f'(u^* - \bar{u} + u_h + v_{01})(\xi(t)v^* + \eta(t)\check{v}^* - \bar{v} + \widehat{v}_h \\ \quad +v_{02}) - f'(u_h)\widehat{v}_h + (\lambda^* - \widehat{\lambda}_h + \widehat{\lambda}_h)(\xi(t)v^* \\ \quad +\eta(t)\check{v}^* - \bar{v} + \widehat{v}_h + v_{02}) - \widehat{\lambda}_h\widehat{v}_h\} \\ \lambda^* - \widehat{\lambda}_h + \int_\Omega(\xi(t)v^* + \eta(t)\check{v}^* - \bar{v} + \widehat{v}_h + v_{02})^2\,\mathrm{d}x - 1 \end{pmatrix} \right\} \\
&\quad + \begin{pmatrix} (I_0 - P_{h0})(-\Delta)^{-1}\{f(u^* - \bar{u} + u_h + v_{01}) - f(u_h)\} \\ (I_0 - P_{h0})(-\Delta)^{-1}\{f'(u^* - \bar{u} + u_h + v_{01})(\xi(t)v^* \\ \quad +\eta(t)\check{v}^* - \bar{v} + \widehat{v}_h + v_{02}) - f'(u_h)\widehat{v}_h + (\lambda^* - \widehat{\lambda}_h + \widehat{\lambda}_h)(\xi(t)v^* \\ \quad +\eta(t)\check{v}^* - \bar{v} + \widehat{v}_h + v_{02}) - \widehat{\lambda}_h\widehat{v}_h\} \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \eta(t)P_{h0}(u^* - \bar{u}) + (1 - \eta(t))P_{h0}(u^* - \bar{u}) \\ \eta(t)P_{h0}(\check{v}^* - \bar{v}) - (1 - \eta(t))P_{h0}\bar{v} + P_{h0}\xi(t)v^* \\ \eta(t)(\lambda^* - \widehat{\lambda}_h) + (1 - \eta(t))(\lambda^* - \widehat{\lambda}_h) \end{pmatrix} \\
&\quad -[I - F'(\rho)]_h^{-1} \left[\begin{pmatrix} \eta(t)P_{h0}(u^* - \bar{u}) \\ \eta(t)P_{h0}(\check{v}^* - \bar{v}) + P_{h0}\xi(t)v^* \\ \eta(t)(\lambda^* - \widehat{\lambda}_h) \end{pmatrix} \right. \\
&\quad - \left. \begin{pmatrix} \eta(t)P_{h0}(-\Delta)^{-1}\{f(u^* - \bar{u} + u_h + v_{01}) - f(u_h)\} \\ \eta(t)P_{h0}(-\Delta)^{-1}\{f'(u^* - \bar{u} + u_h + v_{01})(\check{v}^* - \bar{v} + \widehat{v}_h + v_{02}) \\ \quad -f'(u_h)\widehat{v}_h + (\lambda^* - \widehat{\lambda}_h + \widehat{\lambda}_h)(\check{v}^* - \bar{v} + \widehat{v}_h + v_{02}) - \widehat{\lambda}_h\widehat{v}_h\} \\ \quad +\xi(t)P_{h0}(-\Delta)^{-1}\{f'(u^*)v^* + \lambda^* v^*\} \\ \eta(t)(\lambda^* - \widehat{\lambda}_h) + \eta(t)\left(\int_\Omega(\check{v}^* - \bar{v} + \widehat{v}_h + v_{02})^2\,\mathrm{d}x - 1\right) \end{pmatrix} \right]
\end{aligned}$$

$$
\begin{aligned}
&+\begin{pmatrix}
(\eta(t)+1-\eta(t))(I_0-P_{h0})(-\varDelta)^{-1}\{f(u^*-\bar{u}+u_h+v_{01}) \\
\quad -f(u_h)\} \\
\eta(t)(I_0-P_{h0})(-\varDelta)^{-1}\{f'(u^*-\bar{u}+u_h+v_{01})(\check{v}^*-\bar{v}+\widehat{v}_h) \\
\quad +v_{02}-f'(u_h)\widehat{v}_h+(\lambda^*-\widehat{\lambda}_h+\widehat{\lambda}_h)(\check{v}^*-\bar{v}+\widehat{v}_h+v_{02}) \\
\quad -\widehat{\lambda}_h\widehat{v}_h\}+(1-\eta(t))(I_0-P_{h0})(-\varDelta)^{-1}\{-f'(u_h)\widehat{v}_h-\widehat{\lambda}_h\widehat{v}_h\} \\
\quad +(I_0-P_{h0})(-\varDelta)^{-1}\{f'(u^*)v^*+\lambda^*v^*\}\xi(t) \\
0
\end{pmatrix} \\
&= \eta(t)T(u^*-\bar{u},\check{v}^*-\bar{v},\lambda^*-\widehat{\lambda}_h) \\
&\quad+\begin{pmatrix}
(1-\eta(t))(P_{h0}(u^*-\bar{u})+(I_0-P_{h0})(-\varDelta)^{-1}\{f(u^*)-f(u_h)\}) \\
(\eta(t)-1)P_{h0}\bar{v}+P_{h0}\xi(t)v^*+(1-\eta(t))(I_0-P_{h0})(-\varDelta)^{-1} \\
\quad \times\{-f'(u_h)\widehat{v}_h-\widehat{\lambda}_h\widehat{v}_h\}+(I_0-P_{h0})(-\varDelta)^{-1} \\
\quad \times\{f'(u^*)v^*+\lambda^*v^*\}\xi(t) \\
(1-\eta(t))(\lambda^*-\widehat{\lambda}_h)
\end{pmatrix} \\
&= \begin{pmatrix}
\eta(t)T_1(u^*-\bar{u},\check{v}^*-\bar{v},\lambda^*-\widehat{\lambda}_h)+(1-\eta(t))(u^*-\bar{u}) \\
\xi(t)v^*+\eta(t)\check{v}^*-\bar{v} \\
\eta(t)T_3(u^*-\bar{u},\check{v}^*-\bar{v},\lambda^*-\widehat{\lambda}_h)+(1-\eta(t))(\lambda^*-\widehat{\lambda}_h)
\end{pmatrix}.
\end{aligned}
$$

Here, the second equality follows using (2.7) and (2.8), the identities such that

$$\lambda^*-\widehat{\lambda}_h=\eta(t)(\lambda^*-\widehat{\lambda}_h)+(1-\eta(t))(\lambda^*-\widehat{\lambda}_h),$$

and the fact that $-\bar{u}+v_{01}+u_h=0$ and $-\bar{v}+v_{02}+\widehat{v}_h=0$. Therefore, we can prove that $g(t)-\bar{v}$ is a fixed point of $p_{u^*-\bar{u},\lambda^*-\widehat{\lambda}_h}$ for all $t\in\mathbb{R}$. In particular,

$$g(t_1)=\check{v}^* \quad \text{for } t_1\equiv\sin^{-1}\left(-\frac{\sqrt{1-\kappa}}{\sqrt{2}}\right)$$

holds. Notice that $g(t)$ is continuous in t and not constant around t_1, and that the fixed point of $p_{u^*-\bar{u},\lambda^*-\widehat{\lambda}_h}$ exists in the interior of V by (3.6). Hence, there exists a real number $t^*\neq t_1$ which is sufficiently close to t_1 and satisfies

$$g(t^*)-\bar{v}\neq\check{v} \quad\text{and}\quad g(t^*)-\bar{v}\in V.$$

This contradicts the uniqueness of the fixed point of $p_{u^*-\bar{u},\lambda^*-\widehat{\lambda}_h}$ in V. Consequently, we have

$$v^*=\check{v}^* \quad\text{or}\quad v^*=-\check{v}^*$$

which implies

$$v^*-\bar{v}\in V \quad\text{or}\quad -v^*-\bar{v}\in V.$$

Since both v^* and $-v^*$ are normalized eigenfunctions corresponding to the eigenvalue λ^*, if both $v^*-\bar{v}\in V$ and $-v^*-\bar{v}\in V$ hold, then both $(u^*-\bar{u},v^*-\bar{v},\lambda^*-\widehat{\lambda}_h)$ and $(u^*-\bar{u},-v^*-\bar{v},\lambda^*-\widehat{\lambda}_h)$ are fixed points of T in $U\times V\times\varLambda$. Hence, we have $v^*=-v^*$ by Theorem 1 and therefore $v^*=0$, which is a contradiction. Thus, $v^*-\bar{v}\in V$ and $-v^*-\bar{v}\in V$ cannot hold both. Therefore, we conclude the lemma. □

We next prove two additional lemmas, which are needed for the local uniqueness of eigenfunctions.

For fixed $u \in U$ and $v \in V$, we define the operator $p_{u,v} : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$p_{u,v}(\lambda) \equiv T_3(u, v, \lambda), \tag{3.10}$$

where T_3 is the same as in (3.1). Then (2.18) yields

$$p_{u,v}(\lambda) \in \operatorname{int}(\Lambda) \quad \text{for all } \lambda \in \Lambda. \tag{3.11}$$

We have the following lemma.

Lemma 6. *There exists a fixed point of $p_{u,v}$ in Λ for each $u \in U$, $v \in V$ and it is unique in Λ.*

Proof. In analogy to the proof of Lemma 4, we have, for some $0 < k < 1$, and for any $\lambda_1, \lambda_2 \in \Lambda$

$$|p_{u,v}(\lambda_2) - p_{u,v}(\lambda_1)| \leqslant k|\lambda_2 - \lambda_1|. \tag{3.12}$$

Therefore, we can again apply Banach's fixed point theorem for $p_{u,v}$ to prove the lemma. □

Now, we are able to derive the following lemma.

Lemma 7. *Assume that* (2.18) *in Theorem* 1 *holds and let $(u^* - \bar{u}, v^* - \bar{v}, \lambda^* - \widehat{\lambda}_h)$ be a fixed point of T. If $u^* - \bar{u} \in U$ and $v^* - \bar{v} \in V$ hold, then we have $\lambda^* - \widehat{\lambda}_h \in \Lambda$.*

Proof. We denote a fixed point of $p_{u^*-\bar{u},v^*-\bar{v}}$ in Λ by μ, which is unique in Λ by Lemma 6, and define μ^* by $\mu^* \equiv \mu + \widehat{\lambda}_h$.

Assume that $\lambda^* \neq \mu^*$ holds. Defining $\nu(t)$ for each $t \in \mathbb{R}$ as

$$\nu(t) \equiv (1 - t)\lambda^* + t\mu^*,$$

we can find that $\nu(t) - \widehat{\lambda}_h$ is a fixed point of $p_{u^*-\bar{u},v^*-\bar{v}}$ for all $t \in \mathbb{R}$ by a similar calculation in the proof of Lemma 5.

In particular, we have $\nu(1) = \mu^*$. Since the fixed point of $p_{u^*-\bar{u},v^*-\bar{v}}$ exists in the interior of Λ by (3.11), by the property of $\nu(t)$ there exists a real number $t^{**} \neq 1$ sufficiently close to 1 satisfying

$$\nu(t^{**}) - \widehat{\lambda}_h \neq \mu \quad \text{and} \quad \nu(t^{**}) - \widehat{\lambda}_h \in \Lambda.$$

This contradicts the uniqueness of the fixed point of $p_{u^*-\bar{u},v^*-\bar{v}}$ in Λ. Therefore, we have

$$\lambda^* - \widehat{\lambda}_h = \mu \in \Lambda,$$

and the lemma is proved. □

From Theorem 1 and Lemmas 5 and 7, we can obtain the following main theorem in this section.

Theorem 2. *We assume that a set* $W = U \times V \times \Lambda$ *satisfies the condition in Theorem* 1. *Let* (u^*, v^*, λ^*) *be a unique solution of* (2.5) *such that* $u^* \in \bar{u} + U$, $v^* \in \bar{v} + V$, $\lambda^* \in \widehat{\lambda}_h + \Lambda$ *and set* u *in* (2.4) *as* $u = u^*$. *Then we have*:
(i) v^* *is a unique eigenfunction of* (2.4) *s.t.* $v^* - \bar{v} \in V$ *with* $\|v^*\|_{L^2}^2 = 1$.
(ii) λ^* *is a unique eigenvalue of* (2.4) *s.t.* $\lambda^* - \widehat{\lambda}_h \in \Lambda$.
(iii) (v^*, λ^*) *is a unique eigenpair of* (2.4) *in the set* $(\bar{v} + V) \times (\widehat{\lambda}_h + \Lambda)$.
(iv) *Multiplicity of* λ^* *is* 1.

Proof. (i) The existence of the normalized eigenfunction v^* satisfying $v^* - \bar{v} \in V$ is confirmed by Theorem 1. We now show its uniqueness.

Assume that there exists a normalized eigenfunction $\breve{v}^*$ which is different from v^* and satisfies $\breve{v}^* - \bar{v} \in V$. Let λ^* and μ^* be the eigenvalues corresponding to the eigenfunctions v^* and $\breve{v}^*$, respectively. Then we have $\lambda^* - \widehat{\lambda}_h, \mu^* - \widehat{\lambda}_h \in \Lambda$ by Lemma 7. Therefore, Theorem 1 implies

$$\lambda^* = \mu^*, \quad v^* = \breve{v}^*,$$

which contradicts the assumption.

(ii) Since Theorem 1 assures the existence of the eigenvalue λ^* satisfying $\lambda^* - \widehat{\lambda}_h \in \Lambda$, we need only to prove its uniqueness.

Assume that there exists another eigenvalue μ^* which is distinct from λ^* and satisfies $\mu^* - \widehat{\lambda}_h \in \Lambda$. Then the normalized eigenfunction $\breve{v}^*$ corresponding to μ^* satisfies either $\breve{v}^* - \bar{v} \in V$ or $-\breve{v}^* - \bar{v} \in V$ by Lemma 5. Similarly, the normalized eigenfunction v^* corresponding to λ^* also satisfies either $v^* - \bar{v} \in V$ or $-v^* - \bar{v} \in V$. Hence, we have $\mu^* = \lambda^*$ by Theorem 1, which is a contradiction.

(iii) It is the trivial assertion by Theorem 1.

(iv) Assuming that λ^* is not simple, there exist two normalized eigenfunctions v^* and $\breve{v}^*$ which correspond to λ^* and are linearly independent. We then have both

$$v^* - \bar{v} \in V \quad \text{or} \quad -v^* - \bar{v} \in V$$

and

$$\breve{v}^* - \bar{v} \in V \quad \text{or} \quad -\breve{v}^* - \bar{v} \in V$$

by Lemma 5. Therefore, we can conclude that either $v^* = \breve{v}^*$ or $v^* = -\breve{v}^*$ by Theorem 1. However, this contradicts the linear independent property of v^* and $\breve{v}^*$. □

We next consider the uniqueness of the solution for nonlinear PDE in U. Once we obtain a set $U \times V \times \Lambda$ satisfying the hypothesis in Theorem 1, the local uniqueness of eigenvalue and eigenfunction are assured by Theorem 2. But in order to assure the uniqueness for solution of (2.3) in $U + \bar{u}$, we need another verification. That is, defining

$$F_1(\widetilde{u}) \equiv (-\Delta)^{-1}\{f(\widetilde{u} + u_h + v_{01}) - f(u_h)\}$$

and the corresponding Newton-like operator, say T_1, we have to consider the uniqueness of the fixed point of T_1. If it is verified in U, or in a set including U, we can confirm the local uniqueness for solution of (2.3) in $U+\bar{u}$. We show such a verified example in Section 5.

4. On the minimum eigenvalue

In this section, we consider the index of eigenvalues, especially enclosure of the first eigenvalue.

Now, we assume that the exact solution $u^* = u_h + v_{01} + \widetilde{u}$ $(\widetilde{u} \in U)$ of (2.3) is obtained and consider the linearized eigenvalue problem

$$\begin{aligned} &-\Delta v - f'(u_h + v_{01} + \widetilde{u})v = \lambda v \quad \text{in } \Omega, \\ &v = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{4.1}$$

In the following, assume that there exist functions $f_1,\ f_2$ satisfying

$$f'(u_h + v_{01} + \widetilde{u})\phi = f_1\phi + f_2\phi \qquad \text{s.t. } f_1 \in L^\infty(\Omega), \quad f_2 \in L^2(\Omega).$$

Let $\underline{c}$ be a constant satisfying

$$-f_1 \geqslant \underline{c} \quad \text{a.e. in } \Omega,$$

and choose a positive real value τ satisfying

$$1 > \tau \geqslant \frac{1}{\pi^2}\,|\Omega|^{1/2}\,\|f_2\|_{L^2}.$$

Here $|\Omega|$ stands for the measure of Ω. Note that we can choose such τ provided that h is sufficiently small, since f_2 consist of an error part of u^*. We then consider the eigenvalue problem with constant coefficient:

$$\begin{aligned} &-(1-\tau)\Delta v + \underline{c}v = \mu v \quad \text{in } \Omega, \\ &v = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{4.2}$$

In these preliminaries, we obtain the following theorem.

Theorem 3. *Let λ_i be the* ith *eigenvalue of* (4.1) *and μ_i be the* ith *eigenvalue of* (4.2). *Then the relation*

$$\lambda_i \geqslant \mu_i \quad (i \in \mathbb{N})$$

holds.

Proof. For a function $q \in L^2(\Omega)$ and $u \in H_0^1(\Omega)$, we have

$$\begin{aligned} |(qu,u)_{L^2}| &\leqslant \|q\|_{L^2}\,\|u^2\|_{L^2} \\ &\leqslant \frac{1}{\pi^2}\,|\Omega|^{1/2}\,\|q\|_{L^2}\,\|u\|^2_{H_0^1} \\ &= \frac{1}{\pi^2}\,|\Omega|^{1/2}\,\|q\|_{L^2}(\nabla u, \nabla u)_{L^2}. \end{aligned}$$

Here we used the following imbedding inequality by Talenti [23]:

$$\|u^k\|_{L^2} = \|u\|_{L^{2k}}^k \leqslant (C(k))^k |\Omega|^{1/2} \, \|u\|_{H_0^1}^k,$$

where

$$C(k) = \frac{k}{2\pi}(k-1)^{-1/2k} \left(\sin \frac{\pi}{k}\right)^{1/2}.$$

Thus, we have

$$(qu, u)_{L^2} \leqslant \frac{1}{\pi^2} \, |\Omega|^{1/2} \|q\|_{L^2} (\nabla u, \nabla u)_{L^2}. \tag{4.3}$$

Therefore,

$$\begin{aligned}
&(\nabla v, \nabla v)_{L^2} - (f'(u_h + v_{01} + \widetilde{u})v, v)_{L^2} \\
&\quad \geqslant (\nabla v, \nabla v)_{L^2} + (\underline{c}v, v)_{L^2} - (f_2 v, v)_{L^2} \\
&\quad \geqslant (\nabla v, \nabla v)_{L^2} + (\underline{c}v, v)_{L^2} - \tau(\nabla v, \nabla v)_{L^2} \\
&\quad = (1-\tau)(\nabla v, \nabla v)_{L^2} + (\underline{c}v, v)_{L^2}
\end{aligned}$$

holds. Then the minimum–maximum principle for eigenvalues (cf. [2]) proves the theorem. □

If we can check the condition

$$\widehat{\lambda}_h + \sup\ \Lambda < \mu_2 \tag{4.4}$$

for the interval $\widehat{\lambda}_h + \Lambda$ obtained by the method in Sections 2 and 3, then we can confirm that the unique eigenvalue in $\widehat{\lambda}_h + \Lambda$ is the first eigenvalue by the above theorem. If the inequality (4.4) is not satisfied, a homotopy method (e.g., [16,17]) can be used to obtain some information on indices.

Remark. We cannot obtain the exact eigenvalue of (4.2) for general domain in $\mathbb{R}^2$. In the case of a rectangular domain such as $\Omega = (0, a) \times (0, b)$ $(a, b > 0)$, we have

$$\mu_{n,m} = (1-\tau)\pi^2 \left(\frac{n^2}{a^2} + \frac{m^2}{b^2}\right) + \underline{c} \quad (n, m \in \mathbb{N}).$$

We can obtain similar explicit formulas, e.g., for spherical domains. For more general domains, we need some eigenvalue excluding methods such as described in [8] or one may apply a domain homotopy making use of the domain monotonicity of Dirichlet eigenvalues (cf. [17]).

5. A numerical example

We now give an example whose first eigenvalue has been enclosed by our method. We set the following conditions.

Ω is the rectangular domain $(0,1)\times(0,1)\subset\mathbb{R}^2$, and the interval $(0,1)$ was partitioned into 20 pieces ($h=1/20$) to use rectangular elements. The basis in S_h consists of continuous, piecewise biquadratic polynomials on Ω. ($M=\dim S_h=1521$) The inflation parameter δ in (2.41) is set as 0.0001. Then we can take the constant appearing in Assumption 1 as $C_0=1/2\pi$ [15].

In the calculations, we used interval arithmetic in order to avoid the effects of rounding errors in the floating-point computations. The computations were carried out on a Sun Enterprise 450 using the interval library PROFIL coded by Knüppel of the Technical University of Hamburg–Harburg [5]. PROFIL is known as a portable C++ class fast interval library and two interval solvers provided by Rump [20] are supported.

Example. We treated the case of $f(u)=u^2(u\not\equiv 0)$; i.e., we have enclosed the triple $(u,v,\lambda)\in X$ satisfying

$$\begin{aligned}&(\nabla u,\nabla\phi)_{L^2}=(u^2,\phi)_{L^2}\quad\forall\phi\in H_0^1(\Omega),\\&(\nabla v,\nabla\phi)_{L^2}=(\lambda v+2uv,\phi)_{L^2}\quad\forall\phi\in H_0^1(\Omega),\\&\|v\|_{L^2}^2=1.\end{aligned}\tag{5.1}$$

We calculated a finite element approximate solution $(u_h,\widehat{v}_h,\widehat{\lambda}_h)$ satisfying (2.6) using the interval Newton method. Figs. 1 and 2 display the shape of u_h and $\widehat{v}_h$. The verified results are given in Table 1. In Table 1, we know that there exists an eigenvalue satisfying (2.5) in the interval

$$[\inf\widehat{\lambda}_h-(Y_{2M+3}+Z_{2M+3}),\sup\widehat{\lambda}_h]\subset[-21.11061856,-20.70639871]$$

and that it is unique in the interval

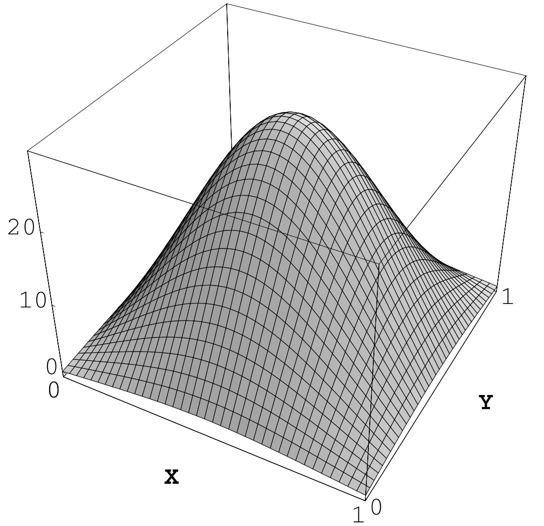

Fig. 1. Approximate solution u_h for nonlinear problem.

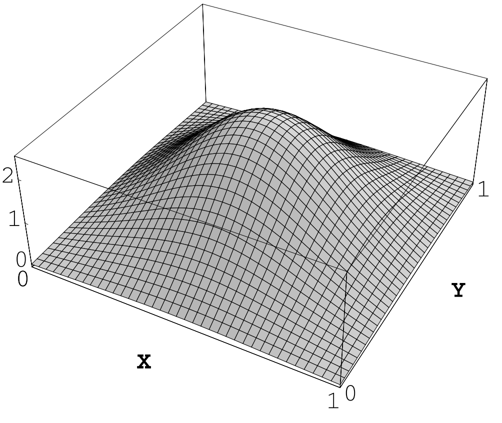

Fig. 2. Approximate eigenfunction $\widehat{v}_h$.

$$[\inf \widehat{\lambda}_h - W_{2M+3}, \sup \widehat{\lambda}_h] \subset [-21.1106288, -20.70639871]$$

by Theorem 2. It can be seen that this eigenvalue is minimum in the set of triple (u, v, λ) satisfying (2.5) by estimating the first and second eigenvalues of (4.2) and by using Theorem 3. Here we used the following splitting:

$$f'(u_h + v_{01} + \widetilde{u})\phi = (2u_h + P_{h0}\widetilde{u})\phi + (2v_{01} + 2(I_0 - P_{h0})\widetilde{u})\phi,$$

that is,

$$f_1 \equiv 2u_h + P_{h0}\widetilde{u}, \qquad f_2 \equiv 2v_{01} + 2(I_0 - P_{h0})\widetilde{u}.$$

In this case, we can obtain τ as 0.0243404. Namely, we can easily obtain $\mu_1 \leqslant -40.12$ and $\mu_2 \geqslant -11.2318$.

Table 1
Results of verification

$\widehat{\lambda}_h$	$\in [-20.706398717^{105}_{626}]$
$\\|u_h\\|_{L^\infty}$	29.443173
$\\|\widehat{v}_h\\|_{L^\infty}$	2.477386
$\\|v_{01}\\|_{H^1}$	0.135007
$\\|v_{01}\\|_{L^2}$	0.001075
$\\|v_{02}\\|_{H^1}$	0.017522
$\\|v_{02}\\|_{L^2}$	0.00014
Iteration number	15
$\max_{1\leqslant i\leqslant 2M}(Y_i + Z_i)$	0.15755958
$\max_{1\leqslant i\leqslant 2M} W_i$	0.15756748
$Y_{2M+1} + Z_{2M+1}$	0.04564149
W_{2M+1}	0.04564382
$Y_{2M+2} + Z_{2M+2}$	0.01445358
W_{2M+2}	0.014454
$Y_{2M+3} + Z_{2M+3}$	0.40421856
W_{2M+3}	0.4042288

As for the uniqueness of solution for nonlinear problem in U, we carried out the verification for the solution of (2.3) by the usual method (e.g., [12–14]) taking the initial set as U in only one iteration. Namely, we could assure that the solution of (2.3) is unique in the set $U + \bar{u}$.

For further reading

For related topics the interested reader is referred to [1,7,10,11,18,19,21,22,26,27].

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